

Core organization of directed complex networks

N. Azimi-Tafreshi,¹ S. N. Dorogovtsev,^{1,2} and J. F. F. Mendes¹

¹*Departamento de Física da Universidade de Aveiro & I3N,
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal*

²*A.F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*

The recursive removal of leaves (dead end vertices) and their neighbors from an undirected network results, when this pruning algorithm stops, in a so-called core of the network. This specific subgraph should be distinguished from k -cores, which are principally different subgraphs in networks. If the vertex mean degree of a network is sufficiently large, the core is a giant cluster containing a finite fraction of vertices. We find that generalization of this pruning algorithm to directed networks provides a significantly more complex picture of cores. By implementing a rate equation approach to this pruning procedure for directed uncorrelated networks, we identify a set of cores progressively embedded into each other in a network and describe their birth points and structure.

PACS numbers: 64.60.aq, 64.60.ah, 02.10.Ox

I. INTRODUCTION

In terms of connectivity, a network, generally, is a set of connected components, which may include a giant connected component containing a finite fraction of vertices and edges (which is well-defined in infinite networks) [1–4]. Studies of random graphs, starting from the first works [5, 6], essentially focused on the connected components, their statistics and structure [7]. A natural generalization of connected components are k -cores, which are obtained by a special pruning procedure, namely the recursive removal of vertices with degrees smaller than k [8, 9]. The k -core structure of complex networks is being studied extensively [10, 11]. On the other hand, other key subgraphs of networks, so-called cores, are far less studied. For an undirected network, the core is a final result of the application of the following pruning algorithm. Remove recursively the leaves (by definition, vertices of degree 1) with their nearest neighbors from the network. Speaking in more detail, at each step, remove a randomly chosen leaf and its neighboring vertex from the network. Continue the process until no leaves remain. The resulting subgraph consists of finite components and, if the mean vertex degree $\langle q \rangle$ exceeds some threshold, of a giant component containing a finite fraction of vertices. It is this giant component that is usually called the core of a network [12]. The cores are related to a wide range of topical problems for networks such as controllability of networks [13], localization on random graphs [14], and some combinatorial optimization problems like maximum matching [15, 16] and minimum vertex cover [17–19].

As a first application, this pruning algorithm was used by Karp and Sipser who intended to find a maximum matching in graphs [15] (at least, some of the maximum matchings), where a matching is a set of edges of a graph which have no common vertices. They proposed an algorithm which recursively selects an edge (i, j) for which vertex i has degree 1. This edge is included into the

matching. Then all the edges incident to vertex j , including the edge (i, j) are removed from the graph, and the process is recursively repeated. If the resulting graph has no leaves (e.g., when the graph is already reduced to the core) choose an edge at random, include it into the matching and remove all the edges incident to the end vertices of this edge. These steps are repeated until all the edges will be removed. The result of this procedure was suggested to be a maximum matching. Although it was shown that the Karp–Sipser algorithm, in general, actually does not generate the maximum matchings [20] it is often used as a preprocessing step for other maximum matching algorithms [21, 22].

For tree-like graphs, after the pruning, a remaining subgraph may consist only of isolated vertices and a giant core. Bauer and Golinelli studied the leaf removal algorithm on the Erdős–Rényi graphs [12], which are tree-like in the infinite network limit. Using generating functions, they showed that there is a continuous phase transition at the mean degree $\langle q \rangle = e$, where $e = 2.718\dots$, so that below the transition the remaining subgraph contains only isolated vertices and above the transition this subgraph consists of isolated vertices and a giant core.

Previous studies of cores were mostly focused on undirected networks, while numerous real-world networks, similarly to the World Wide Web, are directed graphs (digraphs), i.e., their edges are directed. A leaf in directed networks is defined as a vertex of in- or out-degree equal to 1, which gives in- or out-leaves respectively. The result of the recursive removal of these leaves (the algorithm is explained below in detail) is the core of a directed network [13]. Importantly, there were found close relations between the intensively studied controllability of directed networks and the emergence of a core and the problem of maximum matching for directed graphs [13]. These findings stimulated particular interest in the cores in directed networks. Recently, Liu et al. presented analytically derived results for cores in directed random graphs [23]. They defined however the core in a different,

peculiar way. They transformed a directed network into a bipartite graph and then found the core of this graph as if it was undirected unipartite. This convenient approach crucially simplifies the task and allows for a straightforward generating function technique but it actually leaves unsolved the original problem of the core structure of directed networks as it was defined in Ref. [13].

In this paper, we adapt the leaf removal algorithm to the case of digraphs. This extended leaf removal algorithm is introduced in such a way that it fits the definition of matching patterns on digraphs. In digraphs, a matching is a subset of edges such that no two edges in the set share a common starting vertex or a common ending vertex [24]. Similarly to the Karp–Sipser algorithm, used for constructing of a matching on undirected graphs, in the algorithm for digraphs we choose recursively one directed edge (i, j) in which vertex i is leaf. This edge is included in the matching and removed from the graph together with outgoing edges of vertex j , if vertex i is in-leaf, or together with incoming edges of vertex j , if vertex i is out-leaf. This allows one avoiding forbidden configurations in the matching.

The simplicity of the leaf removal algorithm for directed networks enables us to describe the evolution of the structure of the network during the pruning process by applying rate equations. Similar, though significantly more compact, rate equations have been derived for the description of the leaf removal algorithm on undirected graphs, while constructing minimum vertex covers [19, 25]. Using these rate equations we reobtain the size of a core in the configuration model of undirected random networks and the number of links in this core. For the Erdős–Rényi undirected random graphs, we compare our results for cores with those obtained by the generating function technique [12] and demonstrate the preciseness of the rate equation approach. In this work we apply the extended leaf removal algorithm to the configuration model of directed uncorrelated networks with a given degree distribution and derive rate equations which allow us to find the structure of the cores in directed networks. For directed networks with Poisson and power-law degree distributions we compare our results with simulations in Ref. [13] and observe a complete agreement. In- and out-leaves in directed networks can be classified according to the number m of their outgoing and incoming edges, respectively. This enables us to naturally decompose a directed network, layer by layer, removing leaves with higher and higher m , to sub-cores; we name them m -cores. Note that this decomposition of the directed networks differs principally from the well-known k -core decomposition of networks [8].

This paper is organized as follows. In the next section, for the sake of clarity, we first apply the rate equation approach for leaf removal [19, 25] to the core extraction problem for undirected uncorrelated networks and demonstrate the preciseness of this approach. In Sec. III we introduce in detail the extended leaf removal algorithm for directed networks and derive the rate equations.

By using these equations we find the basic characteristics of cores in diverse directed uncorrelated networks and explore the m -core decomposition of directed networks.

II. UNDIRECTED NETWORKS

A. Rate equations

Let us first consider an undirected uncorrelated network with N vertices, L edges and a degree distribution $P(q)$, which is described here by the configuration model [26, 27], i.e., a labelled random graph with a given degree sequence. We apply the leaf removal algorithm [12, 15] to this network. According to this algorithm, at each step we choose at random a leaf and remove it together with its nearest neighbor vertex and all the incident edges to this neighbor. The procedure is iterated until no leaves remain in the network.

The structural evolution of the network during the leaf removal process is described by so-called rate equations for the degree distribution of the remaining network [19, 25]. For the sake of comparison with directed networks, let us remind its meaning (for more detail, see the original derivation in Refs. [19, 25]). Let $N(q, t)$ be the average number of vertices with degree q and $N(t)$ be the total number of remaining vertices at rescaled time $t = T/N$, where T is the number of algorithmic steps. So $\Delta t = 1/N$ as a rescaled time of one iteration. The change of the average number of vertices with degree q after one step is described by the following expression:

$$N(q, t + \Delta t) = N(q, t) - \delta_{q,1} - \frac{qP(q, t)}{\langle q \rangle_t} + \frac{\langle q(q-1) \rangle_t}{\langle q \rangle_t} \frac{(q+1)P(q+1, t) - qP(q, t)}{\langle q \rangle_t}. \quad (1)$$

The delta function in the right-hand side shows that at each step one leaf is removed. In an uncorrelated network, the degree distribution of the end vertices of a uniformly randomly chosen edge is $P^{(1)}(q, t) = qP(q)/\langle q \rangle$. So when we remove the nearest neighbor of the leaf, the number of vertices with degree q , is decreased by 1 with probability $P^{(1)}(q, t)$. After the removal of the leaf and its nearest neighbor with their edges, the number of connections of the second-nearest neighbors of the leaf diminishes. So finally the last term describes the change of $N(q)$ due to the decrease by 1 of the degrees of the second-nearest neighbors of the removed leaf. The number of the second neighbors, in average is equal to the mean degree of the nearest neighbor vertex except one (connection to the leaf), that is, $\sum_q (q-1)P^{(1)}(q, t) = \langle q(q-1) \rangle_t / \langle q \rangle_t$. Note that the last term on the right-hand side of Eq. (1) can be derived strictly implementing a single assumption, namely, the absence correlations in the network during the pruning process.

At each step, the total number of the vertices in the network decreases by 2. The number of removed edges, in

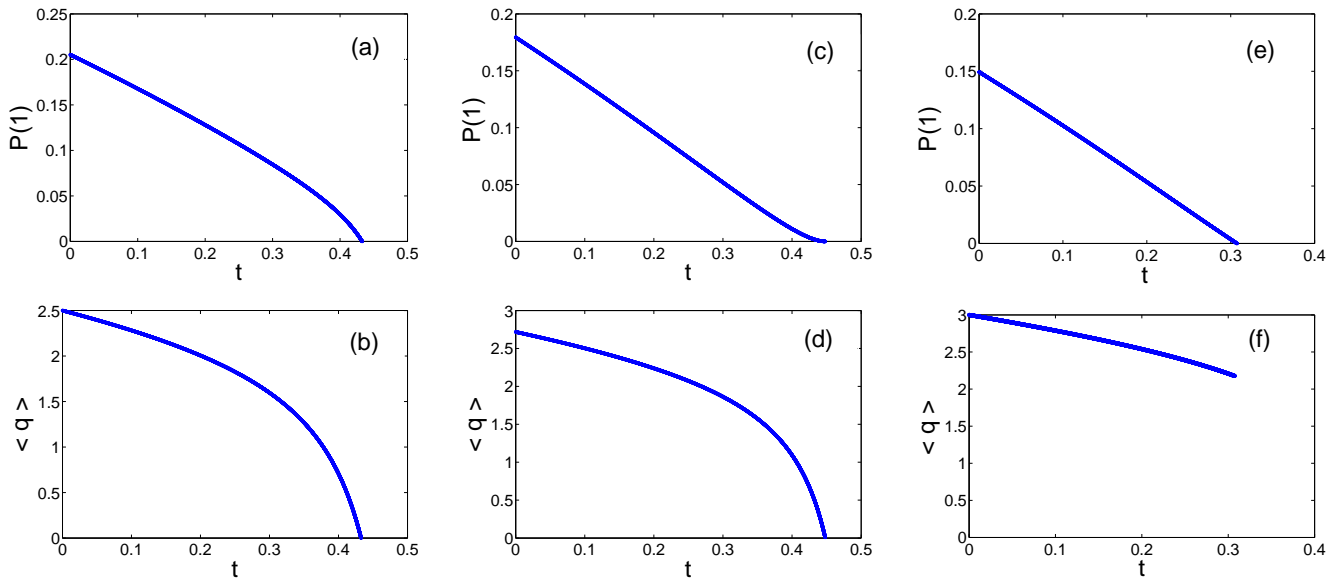


FIG. 1. Different regimes of the evolution of the fraction of leaves $P(1)$ and the corresponding mean degree generated by the leaf removal algorithm applied to the Erdős-Rényi undirected graph with different mean degree c_0 : (a) and (b) $c_0 = 2.5$ is below the critical point, (c) and (d) c_0 takes the critical value equal $e \approx 2.72$, (e) and (f) $c_0 = 3.0$ is above the critical value.

average is equal to the mean degree of the first neighbor of the leaf. This results in the following equations:

$$\frac{\dot{N}(t)}{N} = -2, \quad (2)$$

$$\frac{\dot{L}(t)}{N} = -\frac{\langle q^2 \rangle_t}{\langle q \rangle_t}. \quad (3)$$

Substituting $N(q, t) = P(q, t)N(t)$ into Eq. (1) gives the following evolution equation for the degree distribution [19, 25]:

$$(1 - 2t)\dot{P}(q, t) = 2P(q, t) - \delta_{q,1} - \frac{\langle q^2 \rangle_t}{\langle q \rangle_t^2} q P(q, t) + \frac{\langle q(q-1) \rangle_t}{\langle q \rangle_t^2} (q+1) P(q+1, t). \quad (4)$$

Eq. (4) is actually a set of differential equations, describing the evolution of a network during pruning. These equations are seemingly similar to the rate equations describing the evolution of networks driven by the preferential attachment mechanism [28, 29]. The key difference however is that Eq. (4) is nonlinear due to the moments of the degree distribution $P(q, t)$ which enter the right-hand side of the equation. This nonlinearity makes the solution of the equation particularly difficult. Notably, while deriving, it was assumed that the network remains uncorrelated during this pruning process. This is actually a strong assumption, which was never specially proved. For example, in a well-studied problem of connected components in uncorrelated networks, the giant component generally have degree-degree correlations [30]. This is why it is interesting to compare results for cores obtained

by using the rate equations with those found by other methods.

The set of equations (4) was treated analytically only while studying vertex covers for the Erdős-Rényi random graphs [25]. The analysis was based on the following ansatz. It was assumed that the degree distribution stays Poissonian for $q > 1$ during the whole leaf removal process (though mean degree varies). In the present paper we use a different approach, without making this assumption. We directly solve numerically the set of these equations ($0 \leq q \leq q_{\max}$) and obtain the structure of the core analyzing the evolution of the degree distribution during the pruning process. Note that we consider infinite networks which may have vertices with degrees approaching infinity. In this situation we have to truncate the set of evolution equations.

B. Cores in undirected networks

We apply the leaf removal algorithm to an undirected uncorrelated network with degree distribution $P(q, t=0)$ and a vertex mean degree equal to c_0 , as initial conditions. Algorithm is iterated until no vertices of degree 1 remain. The degree distribution of the network is obtained at each time step by solving Eq. (4). We focus on the evolution of the relative number of leaves, $P(1, t)$. Changing the initial mean degree c_0 , we observe two different behaviors for $P(1, t)$ separated by a critical point, see Fig. 1. Below the critical point, $P(1, t)$ becomes zero exactly in the moment in which the vertex mean degree in the remaining networks reaches zero.

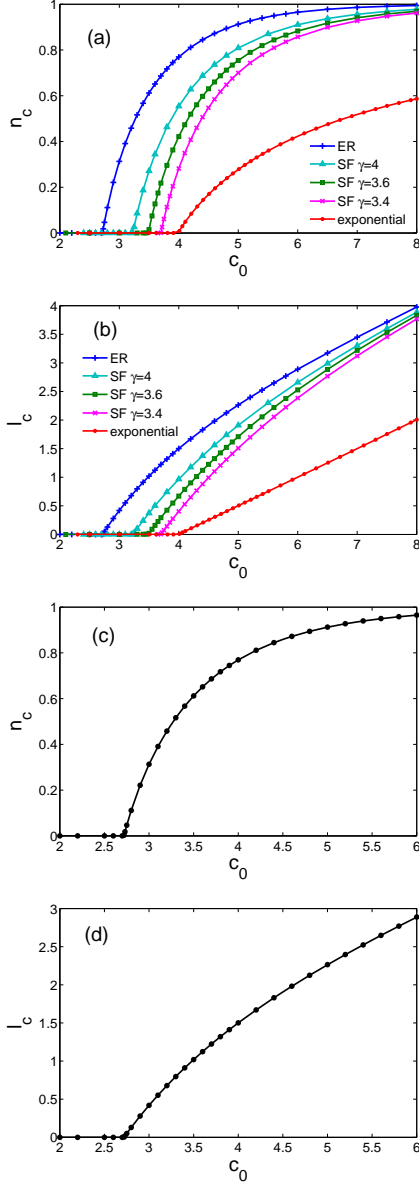


FIG. 2. (a) Relative size $n_c = N_c/N$ and (b) the normalized number of edges $l_c = L_c/N$ in a core for different undirected networks versus the vertex mean degree of an original network. The networks are the Erdős–Rényi random graph (ER), the scale-free networks (SF) with the degree distribution exponents $\gamma = 4, 3.6, 3.4$, and the network with an exponentially decaying degree distribution. (c) and (d) The comparison of our results (dots) for the Erdős–Rényi undirected graphs with the corresponding results obtained by the generating function technique (solid lines) [12].

This means that the algorithm finishes when the network disappears. Contrastingly, above the critical point, $P(1, t)$ becomes zero at a time t^* at which the mean degree is still nonzero. That is, at this moment, there still remains a subgraph with vertices of degrees greater than 1, i.e., the core. The number of remaining vertices except

isolated vertices at time t^* , is the size of the core:

$$\begin{aligned} N_c &= N(t^*) - N(0, t^*) = N(t^*) - N(t^*)P(0, t^*) \\ &= N(1 - 2t^*)[1 - P(0, t^*)]. \end{aligned} \quad (5)$$

The number of edges in the core is obtained from Eq. (3) into which we substitute the solution $P(q, t)$ of Eq. (4), namely,

$$L_c = L(t^*). \quad (6)$$

To demonstrate this approach, we apply it to a few undirected uncorrelated networks. First, consider the Erdős–Rényi random graphs, which have a Poisson degree distribution, $P(q) = e^{-c_0} c_0^q / q!$, in which $c_0 = 2L/N$ is the mean vertex degree for the graph. The rapid decay of the Poisson distribution allows us to reduce greatly the set of the evolution equations which we solve numerically. For the mean degrees c_0 in the range between 1 and 8, it is sufficient to solve the set of the first 21 equations, $q_{\max} = 20$. Figure 1 demonstrates three different regimes of the leaf removal process. Note that at the critical point, $c_0 \approx 2.72 \approx e$, the curve $P(1, t)$ is tangent to the t axis at t^* , see Fig. 1(c). In Fig. 2 we show the relative size of a core, $n_c = N_c/N$ and the normalized number of edges in a core, $l_c = L_c/N$ for the Erdős–Rényi undirected graphs. The points on the plots are obtained in the following way: for each considered value of initial mean degree of the network c_0 , we find the time t^* at which the leaf removal process is completed and the evolution of the distribution $P(q, t \leq t^*)$ and so of its moments, and then the size and the number of edges are calculated from Eqs. (3), (5), and (6). Our approximate value for critical point is $c_0 \approx 2.72$, which agrees with Refs. [12, 23]. Moreover, the comparison of the dependencies of the size and the number of edges in the core on the vertex mean degree c_0 found by the numerical solution of the rate equations and those obtained by using generating functions reveals a complete agreement between these two approaches, see Fig. 2(c) and (d). This suggests that the evolution equations, which were used, are actually asymptotically exact, similarly to the approach of Ref. [12].

We also considered uncorrelated networks with an exponential degree distribution $P(q) = (1 - e^{-1/\kappa})e^{-q/\kappa}$, where $c_0 = e^{-1/\kappa}/(1 - e^{-1/\kappa})$. The critical point is at $c_0 \approx 4.0$ as is shown in Fig. 2. In the case of scale-free networks $P(q) \propto q^{-\gamma}$, n_c and l_c depend on both the γ exponent and the mean degree c_0 . We consider asymptotically scale-free networks with the degree distribution $P(q) = [\frac{c_0(\gamma-2)}{2(\gamma-1)}]^{\gamma-1} \Gamma(q - \gamma + 1, \frac{c_0(\gamma-2)}{2(\gamma-1)}) / \Gamma(q + 1) \cong q^{-\gamma}$, where $\Gamma(s)$ is the gamma function and $\Gamma(s, x)$ is the upper incomplete gamma function [31]. This degree distribution is typical in the static model [32], which is extensively used in simulations. This is why this form will allow for comparison with results obtained in numerical simulations. Here we only explore the region $\gamma > 3$. To obtain the dependencies $n_c(c_0)$ and $l_c(c_0)$ for scale-free networks shown in Fig. 2(a,b) we solve Eq.

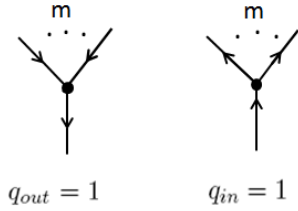


FIG. 3. Various out- and in-leaves in directed networks. For out- and in-leaves, the number m of, respectively, incoming and outgoing edges is arbitrary, $m = 0, 1, 2, \dots$. In the special case of $q_{in} = q_{out} = 1$, the vertex is simultaneously an in- and out-leaf.

(4) with the cutoff $q_{\max} = 100$, ignoring the higher degrees. These dependencies agree with those obtained for cores of these networks within the generating function approach in Ref. [23].

III. DIRECTED NETWORKS

In directed networks, each vertex is characterized by its in-degree q_{in} and out-degree q_{out} , and the joint in,out-degree distribution of a network is $P(q_{in}, q_{out})$. In the directed networks, leaves are defined as vertices with $q_{in} = 1$ or $q_{out} = 1$, see Fig. 3. In general, removing of all leaves leads to a giant core above a critical point. Importantly, the leaf removal algorithm for directed networks should be formulated in such a way that this algorithm will be applicable for combinatorial optimization problems on directed graphs, like the matching problem.

A. Leaf removal algorithm: rate equations

Let us generalize the leaf removal algorithm to directed networks. By definition, the leaves here are vertices with $q_{in} = 1$ and an arbitrary number (including zero) of outgoing edges (in-leaf) or with $q_{out} = 1$ and an arbitrary number (including zero) of incoming edges (out-leaf). In the particular case $q_{in} = q_{out} = 1$, the vertex is simultaneously an in- and out-leaf, but this coincidence does not affect the leaf removal process. The extended leaf removal algorithm is defined in the following way. At each step, one out-leaf or one in-leaf is uniformly randomly chosen with probability $1/2$. The single outgoing edge or incoming of this out- or, respectively, in-leaf is removed, so that this vertex becomes a “normal” (not a leaf) vertex (notice that we do not delete this vertex because it may still have connections with other its neighbors). Next, we focus on the neighbor vertex which was connected with the leaf by that removed edge. If the leaf was an out-leaf, we also remove all its neighbor’s incoming edges; if that leaf was an in-leaf, we remove all its neighbor’s outgoing edges, since they are not allowed to enter in the

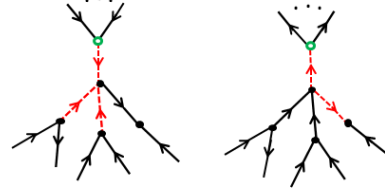


FIG. 4. Leaf removal process on directed networks. The open dots show out- (see left) and in- (see right) leaves. Once a leaf is selected, the dashed edges are removed, and the leaf becomes a normal vertex.

corresponding matching pattern. Figure 4 explains this process. This procedure is iterated until no leaves are remained. Importantly, in contrast to undirected networks, during this process, the number of vertices in the network is conserved. The leaves, being removed, are transformed into normal vertices. Interestingly, in this procedure, a vertex may change its status several times. For example, the following evolution of a vertex is possible. In the original network the vertex is normal, then it becomes an in-leaf, afterwards it is transformed again into a normal vertex, later it becomes an out-leaf, and finally it is an isolated vertex.

Another marked difference from undirected networks is that in directed networks leaves can have various numbers of edges, see Fig. 3. We propose to distinguish and classify leaves in directed networks by the number of their connections. In our leaf removal algorithm we recursively remove leaves having $q_{out} = 1, q_{in} \leq m$ or $q_{in} = 1, q_{out} \leq m$, which results in the specific subgraph which we name m -core. Here m is a given non-negative integer. If m equals the highest vertex in,out-degree in the graph, i.e., $\max(q_{in}, q_{out})$, the algorithm surely removes all possible leaves from a network, which finally results in the main (or central) core or, as we call it here, simply, the core.

Let us derive a master equation, describing the evolution of the structure of an uncorrelated directed network during this algorithm. Let $N(q_{in}, q_{out}, t)$ be the average number of vertices with in-degree q_{in} and out-degree q_{out} at time t . As a consequence of the removal of edges, $N(q_{in}, q_{out}, t)$ is changed after each iteration. The dynamics can be decomposed into three steps.

(1) Choosing a leaf. We choose uniformly at random vertices with degrees $q_{out} = 1, q_{in} \leq m$ (out-leaves) or $q_{in} = 1, q_{out} \leq m$ (in-leaves). At each step we choose one leaf. Suppose that we selected an out-leaf (this choice is made with probability $1/2$). The probability that the in-degree of this leaf turns out to be q_{in} , is equal to $\theta(m - q_{in})P(q_{in}, 1)/[\sum_{q_{in} \leq m} P(q_{in}, 1)]$. Here $\theta(i)$ is defined for integers: $\theta(i \geq 0) = 1$ and $\theta(i < 0) = 0$. We must remove the outgoing edge of this leaf, and so with this probability the number of vertices with $q_{out} = 1, q_{in} \leq m$ decreases by 1 while the number of ver-

tices with $q_{out} = 0, q_{in} \leq m$ increases by 1. In the expression for the difference $N(q_{in}, q_{out}, t + \Delta t) - N(q_{in}, q_{out}, t)$ [see Eq. (7) below], this gives the first two terms on the right-hand side. Similar arguments are valid if the selected leaf is in-leaf. Two corresponding terms can be seen on the right-hand side of Eq. (7).

(2) Changing the degree of that leaf's nearest neighbor which is connected to the leaf by a single incoming or outgoing edge. The probability that this outgoing edge of the out-leaf at its second end has a vertex of degree q_{in}, q_{out} is $P_{out}^{(1)}(q_{in}, q_{out}) = q_{in}P(q_{in}, q_{out})/\langle q_{in} \rangle$. Therefore with this probability the number $N(q_{in}, q_{out})$ is decreased by one, which gives the third term on the right-hand side of Eq. (7). Recall that we remove all incoming edges to this vertex, and so its in-degree changes to $q_{in} = 0$. Consequently, the number of vertices with $q_{in} = 0$ increases by $\delta_{q_{in},0} \sum_{q_{in}} q_{in}P(q_{in}, q_{out})/\langle q_{in} \rangle$,

which is the fourth term on the right-hand side of Eq. (7). A similar argument is valid when we select an in-leaf.

(3) Changing the degree of leaf's second neighbors. When we select an out-leaf and remove its outgoing edge, we also remove all incoming connections from the leaf's neighbor on this edge, see Fig. 4. Consequently, the out-degrees of the corresponding leaf's second neighbors decrease by 1. The number of these incoming edges except the edge attached to the leaf, in average is equal to the mean in-degree of the nearest neighbor of the leaf minus 1, i.e., $\sum_{q_{in}, q_{out}} (q_{in} - 1)P_{in}^{(1)}(q_{in}, q_{out})$. This leads to the fifth term on the right-hand side of Eq. (7). Similar arguments are valid when we select an in-leaf.

Applying all these arguments we finally obtain the evolution equation for the degree distribution $P(q_{in}, q_{out}, t)$:

$$\begin{aligned} N(q_{in}, q_{out}, t + \Delta t) - N(q_{in}, q_{out}, t) &= \dot{P}(q_{in}, q_{out}, t) \\ &= \frac{1}{2} \left(-\frac{\delta_{q_{out},1}\theta(m - q_{in})P(q_{in}, 1, t)}{\sum_{q_{in} \leq m} P(q_{in}, 1, t)} + \frac{\delta_{q_{out},0}\theta(m - q_{in})P(q_{in}, 0, t)}{\sum_{q_{in} \leq m} P(q_{in}, 0, t)} - \frac{q_{in}P(q_{in}, q_{out}, t)}{\langle q_{in} \rangle_t} \right. \\ &\quad + \frac{\delta_{q_{in},0} \sum_{q_{in}} q_{in}P(q_{in}, q_{out}, t)}{\langle q_{in} \rangle_t} + \frac{\langle q_{in}(q_{in} - 1) \rangle_t}{\langle q_{in} \rangle_t} \frac{(q_{out} + 1)P(q_{in}, q_{out} + 1, t) - q_{out}P(q_{in}, q_{out}, t)}{\langle q_{out} \rangle_t} \Big) \\ &\quad + \frac{1}{2} \left(-\frac{\delta_{q_{in},1}\theta(m - q_{out})P(1, q_{out}, t)}{\sum_{q_{out} \leq m} P(1, q_{out}, t)} + \frac{\delta_{q_{in},0}\theta(m - q_{out})P(0, q_{out}, t)}{\sum_{q_{out} \leq m} P(0, q_{out}, t)} - \frac{q_{out}P(q_{in}, q_{out}, t)}{\langle q_{out} \rangle_t} \right. \\ &\quad + \frac{\delta_{q_{out},0} \sum_{q_{out}} q_{out}P(q_{in}, q_{out}, t)}{\langle q_{out} \rangle_t} + \frac{\langle q_{out}(q_{out} - 1) \rangle_t}{\langle q_{out} \rangle_t} \frac{(q_{in} + 1)P(q_{in} + 1, q_{out}, t) - q_{in}P(q_{in}, q_{out}, t)}{\langle q_{in} \rangle_t} \Big). \end{aligned} \quad (7)$$

We used the fact that, no vertices are removed in this process in contrast to undirected networks, i.e., $N(t + \Delta t) = N(t) = N$. The time step is $\Delta t = 1/N$, and in the large network limit we passed from the discrete difference to the time derivative of the degree distribution $P(q_{in}, q_{out}, t) = N(q_{in}, q_{out}, t)/N$. Equation (7) is actually an infinite set of nonlinear differential equations, even more difficult to solve than Eq. (4) for undirected networks.

It turns out possible to express the time derivative of total number of edges of the network in terms of the moments of the degree distribution. If we choose an out-leaf, we remove all incoming edges of the leaf's nearest neighbor (this neighbor is at the second end of the outgoing edge of the leaf). The number of these edges in average is equal to the mean in-degree of the leaf's nearest neighbor vertex. The probability that we reach a vertex of degree q_{in}, q_{out} , following an outgoing edge of the leaf is $P_{out}^{(1)}(q_{in}, q_{out}) = q_{in}P(q_{in}, q_{out})/\langle q_{in} \rangle$. Therefore, the mean in-degree of the leaf's neighbor is $\sum_{q_{in}, q_{out}} q_{in}P_{out}^{(1)}(q_{in}, q_{out}) = \langle q_{in}^2 \rangle_t / \langle q_{in} \rangle_t$. If an in-

degree leaf is chosen, the number of edges which are removed at one step in average is equal to the mean out-degree of the leaf's nearest neighbor, which is, similarly, $\langle q_{out}^2 \rangle_t / \langle q_{out} \rangle_t$. Choosing out-degree or in-degree leaves with equal probability (1/2) leads to the following result:

$$\frac{\dot{L}(t)}{N} = -\frac{1}{2} \left(\frac{\langle q_{in}^2 \rangle_t}{\langle q_{in} \rangle_t} + \frac{\langle q_{out}^2 \rangle_t}{\langle q_{out} \rangle_t} \right). \quad (8)$$

Our numerical results for cores in directed networks are based on Eqs. (7) and (10). One may suggest that they are asymptotically exact, similarly to Eqs. (2)-(4) for undirected uncorrelated networks which we checked, see Figs. 2(c,d).

B. m -core structures

As was explained, we can run the algorithm for different values of m and so arrive at different m -cores. The smallest number m is 0. In this case, only leaves

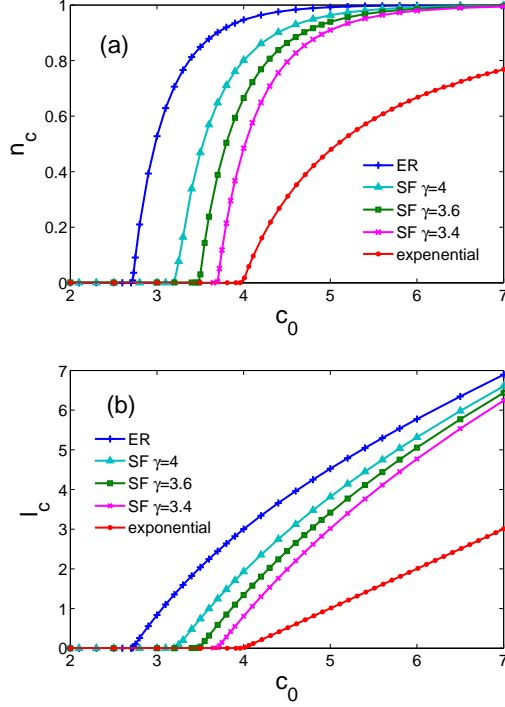


FIG. 5. (a) Relative size and (b) the normalized number of edges of a core versus c_0 (the mean in- or out-degree of a vertex in a network) for different directed networks: the directed Erdős-Rényi graph (ER), the scale-free networks (SF) with the degree distribution exponents $\gamma = 4, 3.6, 3.4$, and the network with an exponentially decaying degree distribution.

with $q_{in} = 1, q_{out} = 0$ and $q_{in} = 0, q_{out} = 1$ are randomly chosen. The leaf removal algorithm is applied to the network until none of these leaves remain. The remaining subgraph is 0-core. If we take $m = 1$, we recursively remove the leaves with $q_{in} = 1, q_{out} = 0, 1$ and $q_{in} = 0, 1, q_{out} = 1$. In this case, we obtain a smaller subgraph within the 0-core, namely the 1-core. Similarly, recursively removing leaves with degrees $q_{in} = 1, q_{out} \leq m$ and $q_{in} \leq m, q_{out} = 1$ we arrive at the m -core (should not be confused with ordinary k -cores). In this way we decompose a directed network into an onion-like structure of m -cores. Although this m -core decomposition of the directed networks differs principally from the well-studied k -core decomposition, they have something in common. In particular, vertices belonging to inner cores, have higher connectivity. This is why the position of a vertex in the system of m -cores may be a better connectivity metrics than the vertex degree which is only a local characteristic.

To find the structure of the m -core of a directed network, we study the evolution the degree distribution of leaves during the leaf removal process. In particular, we must find when all following probabilities become zero: $P(0, 1, t_m^*) = P(1, 1, t_m^*) = P(2, 1, t_m^*) = \dots = P(m, 1, t_m^*) = P(1, 0, t_m^*) = P(1, 1, t_m^*) = P(1, 2, t_m^*) =$

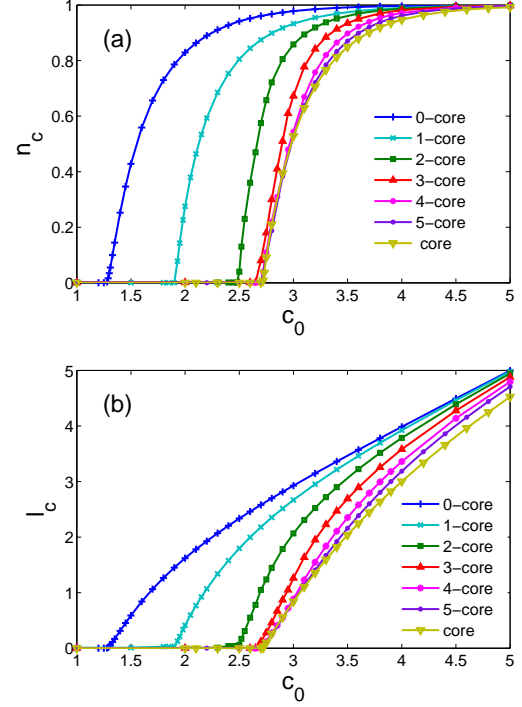


FIG. 6. m -core decomposition for the directed Erdős-Rényi graph. (a) The relative sizes and (b) the normalized numbers of edges of m -cores versus c_0 (which is the mean in- or out-degree of a vertex in a network).

$\dots = P(1, m, t_m^*) = 0$. Our numerical results data for different networks show that $P(1, 0) = P(0, 1)$ is the last probability to become zero, i.e., the leaves with $q_{in} = 1, q_{out} = 0$ and $q_{in} = 0, q_{out} = 1$ are the last to disappear. Hence, to find the time t_m^* at which the pruning process will be completed, we should focus on the evolution of $P(1, 0, t) = P(0, 1, t)$. After we find t_m^* , the size and the number of edges of the m -core can be obtained from the following relations:

$$N_{mc} = N[1 - P(0, 0, t_m^*)], \quad (9)$$

$$L_{mc} = L(t_m^*). \quad (10)$$

At first, let us study removing of all possible leaves from a directed network, which produces the main core of the network (or simply a core). Note that because in our leaf removal algorithm, in- and out-leaves are chosen with equal probability, we can consider only symmetric in, out-degree distributions satisfying $P(q_{in}, q_{out}) = P(q_{out}, q_{in})$. We study networks in which in- and out-degrees of vertices are uncorrelated, i.e., $P(q_{in}, q_{out}) = P(q_{in})P(q_{out})$. We apply the algorithm to a few directed uncorrelated networks with various degree distributions, namely, (1) a directed Erdős-Rényi graph in which both $P(q_{in})$ and $P(q_{out})$ are Poissonian, i.e., $P(q) = e^{-c_0} c_0^q / q!$, (2) scale-free networks with different values of the degree distribution exponent, in which both $P(q_{in})$ and $P(q_{out})$ are dis-

tributed as $P(q) = [\frac{c_0(\gamma-2)}{2(\gamma-1)}]^{-\gamma-1} \Gamma(q-\gamma+1, \frac{c_0(\gamma-2)}{2(\gamma-1)}) / \Gamma(q+1) \cong q^{-\gamma}$, and (3) a directed network for which both $P(q_{in})$ and $P(q_{out})$ follow the same exponential law. In the leaf removal algorithm, which we perform, in- and out-leaves are chosen with equal probability, and so the evolving in,out-degree distribution remain symmetric during the entire removal process. The results, namely, the relative sizes n_c , and the normalized numbers of edges, l_c , of the cores in these networks are shown in Fig. 5. For the directed networks, c_0 denotes the mean in-degree, which is equal to the mean out-degree (half of the total mean degree). Comparing Figs. 2 and 5 one can see that while the critical values of c_0 in undirected and directed networks coincide, the dependencies $n_c(c_0)$ and $l_c(c_0)$ in undirected and directed networks are essentially different from each other. This difference indicates that the problem of cores of directed networks cannot be reduced to that for undirected nets. In Ref. [13], the core was found from numerical simulations of the directed version of the Gilbert ($G_{n,p}$) random graphs of 10^4 vertices, which are equivalent to the corresponding Erdős–Rényi random graph in the infinite network size limit, and for directed scale-free networks (static model) with the same degree distributions as ours. We compared the dependence $n_c(c_0)$ which we obtained for the directed Erdős–Rényi graph and for the directed scale-free network with $\gamma = 4$, see Fig. 5(a), with those in Figs. S9(b) and (d) from the Supplementary material of Ref. [13] and observed a close agreement.

Finally we study the leave removal process involving only leaves with in-degree or out-degree less than or equal to m , which generates m -cores. There is some maximum value of m at which the corresponding m -core coincides with the main core. This number should depend on c_0 . To demonstrate the m -core decomposition we use a directed Erdős–Rényi graph, for which the number of equations in Eq. (7) can be truncated to a small number due to the rapidly decaying Poisson degree distribution. We study the range of c_0 between 0 and 5, choosing $q_{in,max} = q_{out,max} = 20$, and solve numerically $21^2 = 441$ rate equations for each m fixing the parameter c_0 . Figure 6 shows the relative size and the normalized number of edges for each of m -cores. The most right curves in this figure show $n_c(c_0)$ and $l_c(c_0)$ for the main core. One can see that in the range $e < c_0 < 3.0$, the 5-core coincides with the main core. Note that each of m -cores emerges continuously. These phase transitions are of the

second order, and, in the critical region, the m -core's size is proportional to the deviation of c_0 from a critical point.

IV. CONCLUSIONS

In the present paper we applied the rate equation approach to the problem of cores in directed networks. The directedness of edges in these networks results in a wide diversity of leaves, which makes this problem essentially more complex than for undirected networks. We developed the leave removal algorithm involving specific leaves and defined the resulting set of m -cores forming an onion-like structure. Studying the evolution of the degree distribution of various networks during this algorithm, we obtained the basic characteristics of all these cores and the critical points. We suggest that the equations, which we derived for uncorrelated networks, are asymptotically exact. We checked that the result of the application of this approach to the undirected Erdős–Rényi graph completely agrees with the exact result obtained by the generating function technique. We also found that our results for directed networks are in a close agreement with those obtained in simulations.

For the numerical solution of the rate equations we used standard programs for nonlinear differential equations, and, for a given finite q_{max} , our results, in principle, could be obtained with any desired precision. We did not however study scale-free networks with degree distributions exponent γ smaller than, approximately, 3 because of the too large number of the rate equations which we had to solve numerically. The case of $\gamma < 3$ is a challenge for future work.

In summary, we have found an effective way to find the complex structure of cores in directed networks. We suggest that, similarly to cores in undirected graphs, the understanding of cores in directed networks and the proposed m -core decomposition will shed light on a wide range of phenomena in networks.

ACKNOWLEDGMENTS

This work was partially supported by the FCT projects PTDC: FIS/108476/2008, SAU-NEU/103904/2008, MAT/114515/2009, and PEst-C/CTM/LA0025/2011, and FET IP Project MULTIPLEX 317532.

-
- [1] R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).
 - [2] M. E. J. Newman, *SIAM Review* **45**, 167 (2003).
 - [3] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Rev. Mod. Phys.* **80**, 1275 (2008).
 - [4] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, Oxford, 2003); S. N. Dorogovt-

- sev, *Lectures on Complex Networks* (Oxford University Press, Oxford, 2010).
- [5] R. Solomonoff and A. Rapoport, *Bull. Math. Biophys.* **13**, 107 (1951).
- [6] P. Erdős and A. Rényi, *Publicationes Mathematicae Debrecen* **6**, 290 (1959).
- [7] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, *Phys. Rev. E* **64**, 026118 (2001).

- [8] S. B. Seidman, *Social Networks* **5**, 269 (1983).
- [9] B. Bollobás, The evolution of sparse graphs, in B. Bollobás (ed.), *Graph Theory and Combinatorics: Proc. Cambridge Combinatorial Conf. in honour of Paul Erdős*, (Academic Press, New York, 1984), pp. 35-57.
- [10] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Phys. Rev. Lett.* **96**, 040601 (2006).
- [11] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Phys. Rev. E* **83**, 051134 (2011).
- [12] M. Bauer and O. Golinelli, *Eur. Phys. J. B* **24**, 339 (2001).
- [13] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, *Nature* **473**, 167 (2011).
- [14] M. Bauer and O. Golinelli, *Phys. Rev. Lett.* **86**, 2621 (2001).
- [15] R. M. Karp and M. Sipser, *Proc. 22nd Annual IEEE Symp. on Foundations of Computer Science* pp. 364-375 (1981).
- [16] A. K. Hartmann and H. Rieger, Matchings, in *Optimization Algorithms in Physics*, eds. A. K. Hartmann and H. Rieger, (Wiley-VCH, Weinheim, 2001), pp. 227-254.
- [17] M. Weigt and A. K. Hartmann, *Phys. Rev. Lett.* **84**, 6118 (2000).
- [18] M. Weigt and A. K. Hartmann, *Phys. Rev. E* **63**, 056127 (2001).
- [19] M. Weigt and A. K. Hartmann, *Phase Transitions in Combinatorial Optimization Problems*, (Wiley-VCH, Weinheim, 2005).
- [20] J. Aronson, A. M. Frieze, and B. Pittel, *Rand. Struct. Alg.* **12**, 111 (1998).
- [21] J. Magun, *ACM J. Exper. Algorith.* **3**, 6 (1998).
- [22] M. Dietzfelbinger, H. Peilke, and M. Rink, *Lecture Notes Computer Sci.* **7276**, 148 (2012).
- [23] Y.-Y. Liu, E. Csoka, H. Zhou, and M. Posfai,
- [24] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications* (Springer-Verlag, Berlin, 2007). arXiv:1206.2550.
- [25] M. Weigt, *Eur. Phys. J. B* **28**, 369 (2002).
- [26] E. A. Bender and E. R. Canfield, *J. Combin. Theor. A* **24**, 296 (1978).
- [27] B. Bollobás, *Eur. J. Combin.* **1**, 311 (1980).
- [28] P. L. Krapivsky, S. Redner, and F. Leyvraz, *Phys. Rev. Lett.* **85**, 4629 (2000).
- [29] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, *Phys. Rev. Lett.* **85**, 4633 (2000).
- [30] P. Bialas and A. K. Oleś, *Phys. Rev. E* **77**, 036124 (2008).
- [31] M. Catanzaro and R. Pastor-Satorras, *Eur. Phys. J. B* **44**, 241 (2005).
- [32] K. -I. Goh, B. Kahng, and D. Kim, *Phys. Rev. Lett.* **87**, 278701 (2001).